Transportation Problems

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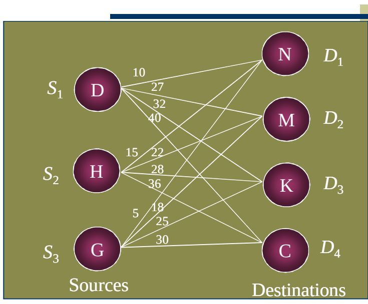
Lecture Outline

- What is a TP?
- Main Concept of TP
- Solution Methods
  - North West Corner Method
  - Least Cost Method
  - VAM Method
- Special Cases of TP
- Numerical Examples

An Example

- A Medical Supply company supplies packets containing drugs & other medical products
- It has three production plants at Dehradun, Haridwar & Gurgaon.
- It has four distribution warehouses at New Delhi, Mumbai, Kolkata & Chennai.
- The packets are distributed directly to warehouses from the plants.
- The table on the next slide shows the costs per pack to ship to the four warehouses.

Network Representation of TP



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PLANT	WAREHOUSE New Delhi	Mumbai	Kolkata	Chennai
Dehradun	10	27	32	40
Haridwar	15	22	28	36
Gurgaon	05	18	25	30

Prod. Capacity	
Dehradun	400
Haridwar	300
Gurgaon	500
-----------	1200

Demand	
New Delhi	375
Mumbai	250
Kolkata	350
Chennai	225
--------	1200

Continued…

From Plant	TO WAREHOUSE N	M	K	C	Plant Capacity
D	10	27	32	40	400
H	15	22	28	36	300
G	05	18	25	30	500
Warehouse Demand	375	250	350	225	1200

Definition

The problem deals with transporting a product manufactured at different plants/factories, known as supply origins, to a number of different warehouses, known as demand destinations. The objective is to satisfy the destination requirements at the minimum transportation cost.

Main Concept

- **Inputs**
  - Availability at sources
  - Requirement at destinations
  - Unit transportation cost from various sources to destinations
- **Objective**

  To determine transportation schedule **to minimize** total transportation cost

Assumptions in TP

1. The per item shipping cost remains constant, regardless of the number of units shipped.
2. All the shipping from the sources to the destinations occur simultaneously. No waiting is allowed.
3. The total supply is equal to the total demand.

Mathematical Model

- Define c_{ij} as the cost to ship one unit from supply origin i to demand destination j.
- Demand at location j is D_j.
- Supply at origin i is S_i
- x_{ij} is the quantity shipped from supply origin i to demand destination j.

Continued…

	Dest 1	Dest 2	…	Dest n	Available Supply
Origin 1	c_{11}	c_{12}	…	c_{1n}	S_1
Origin 2	c_{21}	c_{22}	…	c_{2n}	S_2
…	…	…	…	…	…
Origin m	c_{m1}	c_{m2}	…	c_{mn}	S_m
Demand	D_1	D_2	…	D_n	Total Demand = Total supply

Continued…

x_{11}	x_{12}	…	x_{1n}	S_1
x_{21}	x_{22}	…	x_{2n}	S_2
…	…	…	…	…
x_{m1}	x_{m2}	…	x_{mn}	S_m
D_1	D_2	…	D_n	

Mathematical Model: TP as an LPP

$$\min: \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} x_{ij} = S_i, \quad \text{for } i = 1, \dots, m$$

$$\sum_{i=1}^{m} x_{ij} = D_j, \quad \text{for } j = 1, \dots, n$$

$$x_{ij} \geq 0, \quad \text{for all } i, j$$

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An Important Result

- In a TP, the basic feasible solution will contain at most *(m + n – 1)* positive variables.

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Methods for Solving TP

- North West Corner Method
- Least Cost Method
- VAM Method

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How to solve a TP?

1. Define the objective function to be minimized.
2. Set up a transportation table with *m* rows representing the supply origins and *n* columns representing the demand destinations.
3. Develop an initial feasible solution to the problem by any of these methods
   (a) The North west corner rule
   (b) Lowest cost entry method
   (c) Vogel's approximation method.

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Continued…

4. Examine whether the initial solution is feasible or not (if the solution has allocations in (m+n-1) cells with independent positions).

5. Test whether the solution obtained in the above step is optimum or not.

6. If the solution is not optimum, modify the shipping schedule. Repeat the above until an optimum solution is obtained.

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North-West Corner Method

1. Select the northwest corner cell of the transportation table and allocate as many units as possible [minimum between available supply and demand requirements i.e., (min (S_1, D_1)].
2. Adjust the supply and demand numbers in the respective rows and columns allocation.
3. If the supply for the first row is exhausted, then move down to the first cell in the second row and first column and go to step 2.
4. If the demand for the first column is satisfied, then move horizontally to the next cell in the second column and first row and go to step 2.

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Continued…

5. If for any cell, supply equals demand, then the next allocation can be made in cell either in the next row or column.
6. Continue the procedure until the total available quantity is fully allocated to the cells as required.

Advantages: It is simple and reliable. Easy to compute, understand and interpret.

Disadvantages: This method does not take into considerations the shipping cost, consequently the initial solution obtained by this method require improvement.

Northwest Corner Method

From \ To	D_1	D_2	D_3	D_4	Capacity
S_1	19	7	3	21	100
S_2	15	21	18	6	300
S_3	11	14	15	22	200
Demand	150	100	200	150	600

From \ To	D_1	D_2	D_3	D_4	Capacity
S_1	100 [19]	[7]	[3]	[21]	100
S_2	[15]	[21]	[18]	[6]	300
S_3	[11]	[14]	[15]	[22]	200
Demand	150	100	200	150	600

Start in the upper left-hand corner, "northwest corner" of the schedule and place the largest amount of capacity and demand available in that cell.

From \ To	D_1	D_2	D_3	D_4	Capacity
S_1	100 [19]	[7]	[3]	[21]	100
S_2	50 [15]	[21]	[18]	[6]	300
S_3	[11]	[14]	[15]	[22]	200
Demand	150	100	200	150	600

Since capacity of S_1 is exhausted, move down to repeat the process for the S_2 to D_1 cell. S_2 has sufficient capacity but D_1 can only take 50 packs.

From \ To	D_1	D_2	D_3	D_4	Capacity
S_1	100 [19]	[7]	[3]	[21]	100
S_2	50 [15]	100 [21]	150 [18]	[6]	300
S_3	[11]	[14]	50 [15]	150 [22]	200
Demand	150	100	200	150	600

Now move to the next cells to the right and assign capacity for S_2 to demand until depleted. Then move down to the S_3 row and repeat the process.

Source: Adapted from Lapin, 1994

From \ To	D_1	D_2	D_3	D_4	Capacity
S_1	[19] 1900	[7]	[3]	[21]	100
S_2	[15] 750	[21] 2100	[18] 2700	[6]	300
S_3	[11]	[14]	[15] 750	[22] 3300	200
Demand	150	100	200	150	600

1900
750
2100
2700
750
3300

C =11,500

Multiply the quantity in each cell by the cost.

Example 1

Obtain initial solution of the following transportation problem by using Northwest corner rule method

To / From	D_1	D_2	D_3	D_4	Supply
O_1	1	2	1	4	***30***
O_2	3	3	2	1	***50***
O_3	4	2	5	9	***20***
Demand	***20***	***40***	***30***	***10***	

Continued…

To / From	D_1	D_2	D_3	D_4	Supply
O_1	[20] 1	[10] 2	1	4	***30***
O_2	3	[30] 3	[20] 2	1	***50***
O_3	4	2	[10] 5	[10] 9	***20***
Demand	***20***	***40***	***30***	***10***	

Least Cost Method

1. Select the cell with the lowest transportation cost among all the rows or column of the transportation table.
2. If the minimum cost is not unique, then select arbitrarily any cell with this minimum cost (preferably where maximum allocation is possible).
3. Repeat steps 1 and 2 for the reduced table until the entire supply at different factories is exhausted to satisfy the demand at different warehouses.

Example 2

Obtain initial solution in the following transportation problem by using Least Cost method

To / From	D_1	D_2	D_3	D_4	Supply
O_1	1	2	1	4	***30***
O_2	3	3	2	1	***50***
O_3	4	2	5	9	***20***
Demand	***20***	***40***	***30***	***10***	

Continued…

To / From	D_1	D_2	D_3	D_4	Supply
O_1	1	2	[30] 1	4	***30***
O_2	[20] 3	[20] 3	2	[10] 1	***50***
O_3	4	[20] 2	5	9	***20***
Demand	***20***	***40***	***30***	***10***	

Vegel's Approximation Method (VAM)

1. Compute a penalty for each row and column in the transportation table. The penalty for a given row and column is merely the difference between the smallest cost and next smallest cost in that particular row or column.

2. Identify the row or column with the largest penalty. In this identified row or column, choose the cell which has the smallest cost and allocate the maximum possible quantity to the lowest cost cell in that row or column so as to exhaust either the supply at a particular source or satisfy demand at warehouse (if a tie occurs in the penalties, select that row/column which has minimum cost. If there is a tie in the minimum cost also, select the row/column which will have maximum possible assignments).

Continued…

3. Reduce the row supply or the column demanded by the assigning to the cell.
4. If the row supply is now zero, eliminate the row, if the column demand is now zero, eliminate the column, if both the row supply and the column demand are zero, eliminate both the row and column.
5. Re-compute the row and column difference for the reduced transportation table, omitting rows or columns crossed out in the preceding step.
6. Repeat the above procedure until the entire supply at factories are exhausted to satisfy demand at different warehouses.

Example 3

Obtain initial solution of the following transportation problem by using Vogel's Approximation method

To / From	D_1	D_2	D_3	D_4	Supply
O_1	1	2	1	4	***30***
O_2	3	3	2	1	***50***
O_3	4	2	5	9	***20***
Demand	***20***	***40***	***30***	***10***	

Continued…

	D_1	D_2	D_3	D_4	Supply	
O_1	[20] 1	2	[10] 1	4	***30***	(0) (0) (0) (1) *
O_2	3	[20] 3	[20] 2	[10] 1	***50***	(1) (1) (1) (1) (1)
O_3	4	[20] 2	5	9	***20***	(2) (2) * * *
Demand	***20***	***40***	***30***	***10***		
	(2) (2) (2) * *	(0) (0) (1) (1) (1)	(1) (1) (1) (1) (1)	(3) * * * *		

Optimality Test

The Modified Distribution Method

1. Determine an initial basic feasible solution consisting of *(m + n -1)* allocations using any of the three methods.
2. Determine a set of numbers u_i (i= 1,2,..m) for each row and v_j (j = 1,2..n) for each column. Calculate $c_{ij} = (u_i + v_j)$ for occupied cells. In practice, this is done by choosing the u_i or v_j value as zero corresponding to the row or column containing maximum number of occupied cells.
3. Compute the opportunity cost

$\Delta_{ij} = C_{ij} - (U_i + V_j)$ for each unoccupied cells.

Continued…

4. Check the sign of each opportunity cost:
   If all the Δ_{ij} are positive or zero, then the solution is optimum. If one of the values is zero then there exists an alterative solution for the same transportation cost. If any value is negative, then the given solution is not optimum & further improvement is possible.
5. Select the unoccupied cell with the largest negative opportunity cost as the cell to be included in the next solution.
6. Draw a closed path or loop for the unoccupied cells selected in step 5. It may be noted that right angle turns in this path are permitted only at occupied cells and at the unoccupied cells.

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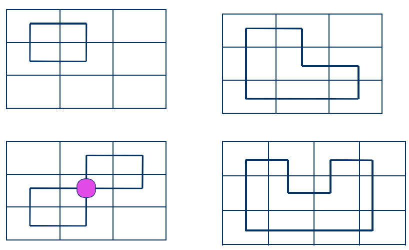

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7. Assign alternative plus and minus signs at the unoccupied cells on the corner points of the closed path with a plus sign at the cell being evaluated.

8. Determine the maximum number of units that should be shipped to this unoccupied cell. The smallest one with a negative sign on the closed path indicates the number of units that can be shipped to the entering cell. This quantity is added to all the cells on the path marked with plus sign and subtracted from those cells marked with minus sign. In this way, the unoccupied cell under consideration becomes an occupied cell making one of the occupied cells as unoccupied cell.

9. Repeat the whole procedure until an optimum solution is found i.e., Δ_{ij} are positive or zero. Finally, calculate new transportation cost.

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Example 3: Continued…

	D_1	D_2	D_3	D_4	Supply	u_i
O_1	[20] 1	2	[10] 1	4	***30***	-1
O_2	3	[20] 3	[20] 2	[10] 1	***50***	0
O_3	4	[20] 2	5	9	***20***	-1
Demand	***20***	***40***	***30***	***10***		
v_j	2	3	2	1		

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Continued…

	D_1	D_2	D_3	D_4	u_i
O_1	[20] 1	(0) 2	[10] 1	(4) 4	-1
O_2	(1) 3	[20] 3	[20] 2	[10] 1	0
O_3	(3) 4	[20] 2	(4) 5	(9) 9	-1
v_j	2	3	2	1	

Here all Δ_{ij} are greater than or equal to zero. Hence the solution is optimum. Optimum solution is $x_{11} = 20$, $x_{13} = 10$, $x_{22} = 20$, $x_{23} = 20$, $x_{24} = 10$, $x_{32} = 20$ & minimum cost is 120.

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Example 4

Solve the following transportation problem by using Vogel's Approximation method:

To / From	D_1	D_2	D_3	Supply
O_1	6	8	4	***14***
O_2	4	9	3	***12***
O_3	1	2	6	***5***
Demand	***6***	***10***	***15***	31

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Continued…

				Supply	u_i
	[1] 6	[10] 8	[3] 4	***14***	0
	(-1) 4	(2) 9	[12] 3	***12***	-1
	[5] 1	(-1) 2	(7) 6	***5***	-5
Demand	***6***	***10***	***15***		
v_j	6	8	4		

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Continued…

[1] 6	[10] 8	[3] 4
(-1) 4	9	[12] 3
[5] 1	2	6

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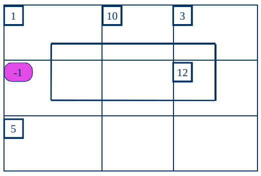
Continued…



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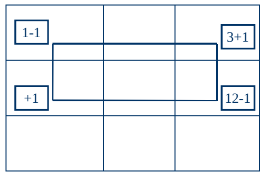

Min{1, 12} = 1

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Continued…

				Supply	u_i
	(1) 6	[10] 8	[4] 4	***14***	0
	[1] 4	(2) 9	[11] 3	***12***	-1
	[5] 1	(-2) 2	(6) 6	***5***	-4
Demand	***6***	***10***	***15***		
v_j	5	8	4		

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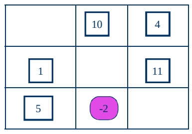
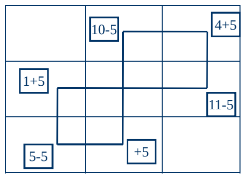
Continued…





Min{5, 10} = 5

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Continued…

				Supply	u_i
	(1) 6	[5] 8	[9] 4	***14***	1
	[6] 4	(2) 9	[6] 3	***12***	0
	(2) 1	[5] 2	(8) 6	***5***	-5
Demand	***6***	***10***	***15***		
v_j	4	7	3		

$x_{12} = 5$, $x_{13} = 9$, $x_{21} = 6$, $x_{23} = 6$, $x_{32} = 5$, Min cost = 128

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Degeneracy in TP

- If at every initial stage or in any subsequent iteration the number of allocations be less than *(m + n – 1)* then we have a case of degeneracy.
- To resolve this case, we allocate a very small positive quantity epsilon to one or more (as many required) of the empty cells (lowest cost cells) and consider these cells to be the occupied cells. With this choice of epsilon, the original problem is not changed. Epsilon is considered as a real positive allocation as long as is required. Finally, it will be omitted.

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Example 5

Solve the following transportation problem by using Vogel's Approximation method:

To / From	D_1	D_2	D_3	Supply
O_1	8	7	3	***60***
O_2	3	8	9	***70***
O_3	11	3	5	***80***
Demand	***50***	***80***	***80***	

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Continued…

	D_1	D_2	D_3	Supply
O_1	8	7	[60] 3	***60***
O_2	[50] 3	8	[20] 9	***70***
O_3	11	[80] 3	5	***80***
Demand	***50***	***80***	***80***	

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Continued…

- We now add a small positive quantity ε to a cell such that this does not result in forming a loop among some or all of the occupied cells and make them dependent. For a dependent set of cells unique determination of u_i and v_j will not be possible.

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Continued…

	D_1	D_2	D_3	Supply	u_i
O_1	(11) 8	[ε] 7	[60] 3	***60***	0
O_2	[50] 3	(-5) 8	[20] 9	***70***	6
O_3	(18) 11	[80] 3	(6) 5	***80***	-4
Demand	***50***	***80***	***80***		
v_j	-3	7	3		

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Continued…

(11)	[ε]	[60]
[50]	(-5)	[20]
(18)	[80]	(6)

	$\varepsilon - \varepsilon$	$60 + \varepsilon$
	$+\varepsilon$	$20 - \varepsilon$

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Continued…

(11) 8	(5) 7	[60] 3	-6
[50] 3	[ε] 8	[20] 9	0
(13) 11	[80] 3	(1) 5	-5
3	8	9	

Here all Δ_{ij} are greater than zero. Hence the solution is optimum. Optimum solution is $x_{13} = 60$, $x_{21} = 50$, $x_{23} = 20$, $x_{32} = 80$ & minimum cost is 750.

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Special Cases in TP

- Unbalanced TP
- Maximizing TP
- No allocation in a particular cell
- Some positive allocation in a particular cell

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Unbalanced TP

In a TP, if the sum of available resources is not equal to sum of demands then the TP is known as an unbalanced TP.

$$\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$$

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Adding Fictitious Source

	A	B	C	S
1	10	2	9	50
2	25	12	6	30
D	20	50	40	

	A	B	C	S
1	10	2	9	50
2	25	12	6	30
3	0	0	0	30
D	20	50	40	

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Adding Fictitious Destination

	X	Y	Z	S
1	15	8	7	80
2	20	3	4	50
D	30	30	40	

	X	Y	Z	T	S
1	15	8	7	0	80
2	20	3	4	0	50
D	30	30	40	30	

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Example

Solve the following unbalanced TP:

	D_1	D_2	D_3	S
O_1	4	3	2	10
O_2	1	5	0	13
O_3	3	8	6	12
D	8	5	4	

(Minimum Cost = 23 units)

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Maximizing TP

A maximizing TP is first converted into a minimizing TP by subtracting all the costs from the highest cost involved in the cost matrix. Then the problem is solved as usual.

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Example

Solve the following maximizing TP:

	D_1	D_2	D_3	S
O_1	8	6	5	150
O_2	6	6	6	150
O_3	10	8	4	150
O_4	8	6	4	150
D	200	200	200	

(Maximum Cost = 3300 units) Alternate optimum exits.

No allocation in a particular cell

If there is no allocation in a particular cell then it indicates that the rout corresponding to that particular cell from the source to destination is prohibited. In such a case, a large positive cost is assigned to that cell so that no allocation will be made on that cell. The solution procedure remains same.

Some positive allocation in a particular cell

If it is necessary to allocate a given quantity in a particular cell then adjust the availability & requirements accordingly & solve the remaining part of the problem in the usual manner.

Example

Solve the following TP. Given that 8 units must be assigned to (3, 3) cell.

	D_1	D_2	D_3	S
O_1	2	3	11	40
O_2	9	6	7	50
O_3	1	5	4	30
O_4	3	12	2	30
D	50	50	50	

	D_1	D_2	D_3	S
O_1	2	3	11	40
O_2	9	6	7	50
O_3	1	5	[8] 4	22
O_4	3	12	2	30
D	50	50	42	