Simplex Method

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What is the Simplex Method?

- A method (algorithm) to solve linear programs
  - Analogies
    - Gaussian elimination is a method for solving a system of linear equations
    - There are various ways to do sorting and searching with data
- Why the Simplex method?
  - Very efficient in practice
  - Easy to implement
  - Time proven

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The Simplex Method

- When decision variables are more than 2, it is always advisable to use Simplex Method to avoid lengthy graphical procedure.
- The simplex method is not used to examine all the feasible solutions.
- It deals only with a small and unique set of feasible solutions, the set of vertex points (i.e., extreme points) of the convex feasible space that contains the optimal solution.

The simplex method is a method for searching corner point feasible solutions

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Procedure of Simplex Method

- Locate an extreme point of the feasible region
- Examine each boundary edge intersecting at this point to see whether movement along any edge increases the value of the objective function
- If the value of the objective function increases along any edge, move along this edge to the adjacent extreme point. If several edges indicate improvement, the edge providing the greatest rate of increase is selected
- Repeat second & third steps above until movement along any edge no longer increases the value of objective function

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Geometric Interpretation

- The Simplex method starts at a corner-point feasible solution (usually the origin) and moves to adjacent CP solutions along edges until an optimal solution is found.
- A CPF solution is connected to adjacent CPF solutions by edges
- In a problem with n decision variables, each CPF solution will have *n* edges emanating from it

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Continued…

- A CPF solution lies at the intersection of n constraint boundaries.
- By relaxing a single constraint boundary equation (i.e. equality for the constraint is relaxed) an edge is defined which leads to an adjacent CPF solution.
- The simplex method defines
  - How to choose which edge to traverse.
  - When to stop or how to determine that a CPF solution is optimal

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Identifying Solution Type Using Simplex Method

- If there is a bounded feasible region
  - If there is exactly one optimal solution, then it can find the solution efficiently
  - If there are multiple optimal solutions, then it can identify the case & can find all solutions
- If no bounded feasible region exists – can identify
  - Unbounded feasible region
  - No feasible region

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Terminology

- Basic Variable
- Non-basic Variable
- Slack Variable
- Surplus Variable
- Artificial Variable

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LPP in Vector Notations

Maximize $Z = cx$

subject to

$Ax = b, x \geq 0$

where $c = (c_1, c_2, \ldots, c_j, \ldots, c_n)$

$x = (x_1, x_2, \ldots, x_j, \ldots, x_n)^T$

$A = (a_1, a_2, \ldots, a_j, \ldots, a_n)$

$b = (b_1, b_2, \ldots, b_r, \ldots, b_m)^T$

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Notations

a_j = a column vector whose elements are coefficients of x_j in A

B = initial basis containing m-columns of A

x_B = initial basic feasible solution

b = requirement vector

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Objective Value & Z_j

- Objective value

$Z = c_B x_B = c_{B1} x_{B1} + c_{B2} x_{B2} + \ldots + c_{Bm} x_{Bm}$

- The Quantity z_j

$z_j = c_B a_j = c_{B1} a_{1j} + c_{B2} a_{2j} + \ldots + c_{Bm} a_{mj}$

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Net Evaluation: Characteristics

- Net evaluation

$z_j – c_j = c_B a_j - c_j = c_{B1} a_{1j} + c_{B2} a_{2j} + \ldots + c_{Bm} a_{mj} – c_j$

- Optimality condition: $z_j - c_j \geq 0$
- Unbounded solution: $z_j - c_j \leq 0$ and $a_{ij} \leq 0$

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Continued…

- Alternate optimal solution

  z_j - c_j = 0 for some non-basic variable & $a_{ij} \geq 0$ for at least one *i*

- No feasible solution

  If at any stage of the simplex method the optimality condition is satisfied and still at least one artificial variable remains in the basis at the positive level, then the LPP has no feasible solution.

Entering & Departing Vector Rules
(maximization problem)

- Entering Vector Rule (1):

  x_k will enter the basis based on the condition:

  z_k - c_k = Min_j { z_j - c_j | z_j - $c_j \leq 0$ }

- Departing Vector Rule (2):

  x_r will leave the basis based on the condition:

  Min_i { x_{Bi} / a_{ij} | $a_{ij} \geq 0$ }

Initial Simplex Table

		c_j	c_1	c_2	…	c_n
c_B	x_B	b	x_1	x_2	…	x_n
c_{B1}	x_{B1}	b_1	a_{11}	a_{12}	…	a_{1n}
c_{B2}	x_{B2}	b_2	a_{21}	a_{22}	…	a_{2n}
…	…	…	…	…	…	…
c_{Bm}	x_{Bm}	b_m	a_{m1}	a_{m2}	…	a_{mn}
$z_j - c_j$			$z_1 - c_1$	$z_2 - c_2$	…	$z_n - c_n$

$z_j - c_j = c_B\, a_j$ - c_j= $c_{B1}\, a_{1j} + c_{B2}\, a_{2j} + \ldots + c_{Bm}\, a_{mj}$ - c_j

Simplex Algorithm (for a maximization problem)

- Step 1: If the given LPP is of minimization type, then convert it to a maximization type problem.
- Step 2: If any of the components of the requirement vector is negative, multiply the corresponding constraint by (-1) and adjust the direction of inequality. If the constraint is an equality then multiply it by (-1) only.
- Step 3: Add slack, surplus & artificial variables to the constraints according to their requirement.

Continued…

- Step 4: Assign coefficients of slack & surplus variable as 0, and (*-M*) for artificial variables in the changed objective function.
- Step 5: Construct the simplex table by choosing the initial basic feasible solution.
- Step 6: Check the optimality of solution. If all (z_j - $c_j \geq 0$), then the present solution is optimal.
- Step 7: If for at least one a_j , z_j - $c_j \leq 0$ and $a_{ij} \leq 0$ then the problem has unbounded solution and stop there.

Continued…

- Step 8: Choose the entering vector according to Rule 1. In case of tie, chose any one arbitrarily.
- Step 9: Choose the departing vector according to Rule 2. In case of tie, chose any one arbitrarily.
- Step 10: Form the new basis by dropping the departing variable and introducing the new variable. Convert the *key element* to unity and all other elements in its column to zero.
- Step 11: Go to Step 6 and repeat the procedure until either an optimum solution is obtained or there is an indication of unbounded solution.

LPP: Example 1

Max $Z = x_1 + x_2 + 3x_3$

subject to

$3x_1 + 3x_2 + x_3 \leq 3$

$2x_1 + x_2 + 2x_3 \leq 2$

$x_1 , x_2, x_3 \geq 0$

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LPP: Example 1

Max $Z = x_1 + x_2 + 3x_3 + 0.s_1 + 0.s_2$

subject to

$3x_1 + 3x_2 + x_3 + 1.s_1 + 0.s_2 = 3$

$2x_1 + x_2 + 2x_3 + 0.s_1 + 1.s_2 = 2$

$x_1, x_2, x_3, s_1, s_2 \geq 0$

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LPP: Example 1

Table 1

		c_j	1	1	3	0	0	
c_B	x_B	b	x_1	x_2	x_3	s_1	s_2	
0	s_1	3	3	3	1	1	0	
0	s_2	2	2	1	2	0	1	⇒
$z_j - c_j$			-1	-1	-3	0	0	
					↑			

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LPP: Example 1

Table 2

		c_j	1	1	3	0	0
c_B	x_B	b	x_1	x_2	x_3	s_1	s_2
0	s_1	2	2	5/2	0	1	-1/2
3	x_3	1	1	1/2	1	0	1/2
$z_j - c_j$			2	1/2	0	0	3/2

Optimum solution: $x_1 = x_2 = 0, x_3 = 1$, max Z = 3

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LPP: Example 2

Max $Z = 2x_1 + 3x_2$

subject to

$x_1 + x_2 \leq 8$

$x_1 + 2x_2 = 5$

$2x_1 + x_2 \leq 8$

$x_1 , x_2 \geq 0$

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LPP: Example 2

Max $Z = 2x_1 + 3x_2 + 0.s_1 - M.s_2 + 0.s_3$

subject to

$x_1 + x_2 + 1.s_1 + 0.s_2 + 0.s_3 = 8$

$x_1 + 2x_2 + 0.s_1 + 1.s_2 + 0.s_3 = 5$

$2x_1 + x_2 + 0.s_1 + 0.s_2 + 1.s_3 = 8$

$x_1, x_2, s_1, s_2, s_3 \geq 0$

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LPP: Example 2

Table 1

		c_j	2	3	0	-M	0
c_B	x_B	b	x_1	x_2	s_1	s_2	s_3
0	s_1	8	1	1	1	0	0
-M	s_2	5	1	2	0	1	0
0	s_3	8	2	1	0	0	1
$z_j - c_j$			-M-2	-2M-3	0	0	0

LPP: Example 2

Table 2

		c_j	2	3	0	-M	0
c_B	x_B	b	x_1	x_2	s_1	s_2	s_3
0	s_1	11/2	1/2	0	1	…	0
3	x_2	5/2	1/2	1	0	…	0
0	s_3	11/2	3/2	0	0	…	1
$z_j - c_j$			-1/2	0	0	…	0

LPP: Example 2

Table 3

		c_j	2	3	0	-M	0
c_B	x_B	b	x_1	x_2	s_1	s_2	s_3
0	s_1	11/3	0	0	1	…	-1/3
3	x_2	2/3	0	1	0	…	-1/3
2	x_1	11/3	1	0	0	…	2/3
$z_j - c_j$			0	0	0	…	1/3

Optimum solution: $x_1 = 11/3$, $x_2 = 2/3$, max Z = 28/3

Additional Problems

Max $Z = 5x_1 + 3x_2$
s. t. $3x_1 + 5x_2 \leq 15$
$5x_1 + 2x_2 \leq 10$
$x_1, x_2 \geq 0$

($x_1 = 20/19$, $x_2 = 45/19$
Objective value = 235/19)

Max $Z = 5x_1 - 2x_2 + 3x_3$
s. t. $2x_1 + 2x_2 - x_3 \geq 2$
$3x_1 - 4x_2 \leq 3$
$x_2 + 3x_3 \leq 5$
$x_1, x_2, x_3 \geq 0$
($x_1 = 23/3$, $x_2 = 5$, $x_3 = 0$
Objective value = 85/3)

Min $Z = x_1 + x_2 + x_3$
s. t. $x_1 - x_2 + 2x_3 = 2$
$-x_1 + 2x_2 - x_3 = 1$
$x_1, x_2, x_3 \geq 0$
($x_1 = 0$, $x_2 = 4/3$, $x_3 = 5/3$
Objective value = 3)

Min $Z = x_1 - 3x_2 + 2x_3$
s. t. $3x_1 - x_2 + 2x_3 \leq 7$
$-2x_1 + 4x_2 \leq 12$
$-4x_1 + 3x_2 + 8x_3 \leq 10$
$x_1, x_2, x_3 \geq 0$
($x_1 = 4$, $x_2 = 5$, $x_3 = 0$
Objective value = -11)

Continued…

Max $Z = x_1 - x_2 + 2x_3$
s. t. $-x_1 + x_2 + 2x_3 \leq 5$
$-2x_1 + 5x_2 - x_3 \geq 10$
$2x_1 - x_2 + x_3 \geq 4$
$x_1, x_2, x_3 \geq 0$

(Unbounded solution)

Min $Z = x_1 - 2x_2 - 3x_3$
s. t. $-2x_1 + x_2 + 3x_3 = 2$
$2x_1 + 3x_2 + 4x_3 = 1$
$x_1, x_2, x_3 \geq 0$

(No feasible solution)

Negative Variables

In some formulations it makes sense to have negative values allowed for some decision variables (e.g., rate reductions, distance relative to an origin, etc.)

Two cases: Bounded, unbounded

- Bounded

$x_j \geq L_j$ where $L_j < 0$
Replace x_j with x_j' where $x_j' = x_j - L_j$, and $x_j' \geq 0$

- Unbounded

Replace x_j with $x_j^+ - x_j^-$ where x_j^+ and $x_j^- \geq 0$

Examples

Bounded: $x_j >= L_j$ where $L_j < 0$
Replace x_j with x_j' where $x_j' = x_j - L_j$, and $x_j' >=0$

$Z = 3x_1 + 5x_2$

Subject to :

$$\begin{array}{lcrcl} x_1 & & & \le & 4 \\ & & 2x_2 & \le & 12 \\ 3x_1 & + & 2x_2 & \le & 18 \end{array}$$

$x_1 \ge -10, \; x_2 \ge 0$

$Z = 3(x'_1 - 10) + 5x_2$

Subject to :

$$\begin{array}{lcrcl} x'_1 - 10 & & & \le & 4 \\ & & 2x_2 & \le & 12 \\ 3(x'_1 - 10) & + & 2x_2 & \le & 18 \end{array}$$

$x'_1 \ge 0, \; x_2 \ge 0$

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Examples

Unbounded: replace x_j with $x_j^+ - x_j^-$ where x_j^+ and $x_j^- >=0$

$Z = 3x_1 + 5x_2$

Subject to :

$$\begin{array}{lcrcl} x_1 & & & \le & 4 \\ & & 2x_2 & \le & 12 \\ 3x_1 & + & 2x_2 & \le & 18 \end{array}$$

$x_2 \ge 0$

$Z = 3x_1^+ - 3x_1^- + 5x_2$

Subject to :

$$\begin{array}{lcrcl} x_1^+ - x_1^- & & & \le & 4 \\ & & 2x_2 & \le & 12 \\ 3(x_1^+ - x_1^-) & + & 2x_2 & \le & 18 \end{array}$$

$x_1^+ \ge 0, \; x_1^- \ge 0, \; x_2 \ge 0$

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