Network Analysis

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Basic Components

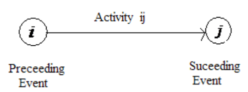
- Collections of interconnected linear forms:
  - Lines
  - Intersections
  - Regions (created by the partitioning of space by the lines)
- Planar (streets, all on same level, vertices at every intersection of edges)
- Non-planar (airline routes, highways with bridges/flyovers, electronic circuits)

Some Applications

- Shortest path between vertices
- Shortest route visiting all locations once and returning to start point (Travelling Salesman Problem)
- Minimum cost of constructing a network
- Identification of zones within specified travel time/cost

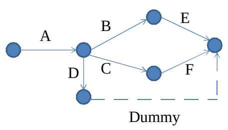
The Other View

- **Network:** It is the graphical representation of a project and is composed of activities that must be completed to reach the end objective of a project, showing the planning sequence of their activities, their dependence and inter-relationships.
- **Event:** An event is a specific physical or intellectual accomplishment in a project plan.
- **Activity:** An activity is a task that consumes resources (time, money etc). It lies between two events.



Continued…

- **Dummy Activity:** Without using any resources, some activities are used to represent a connection between events. This type of activity is known as a dummy activity.



- Purpose of a dummy activity
  - To maintain the proper logic in a network
  - To maintain uniqueness in the numbering system

Goal of Network Analysis

- Solving problems involving networks
- Objectives
  - Maximize efficiency
  - Minimizing time
  - Minimizing expenditure

Rules of Network Construction

1. No event can occur until every activity preceding it has been completed.
2. An activity succeeding an event cannot start until that event has occurred.
3. An event cannot occur more than once.
4. Each activity must start and terminate in an event.
5. Time flows from left to right.
6. An activity must be completed in order to reach the end event.
7. Dummy activities should be used if absolutely necessary.

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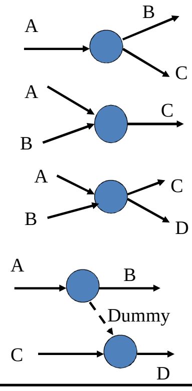
Situations in Network Diagram



A must finish before either B or C can start

both A and B must finish before C can start

both A and C must finish before either of B or D can start

A must finish before B can start

both A and C must finish before D can start

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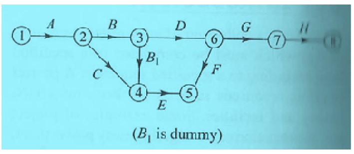
Network Construction (Ex 1)

Draw a network diagram for the following set of activities:

A < B, C; B < D, E; C < E; E < F; D, F < G; G < H

The notation 'X < Y' implies that the activity 'X' must be finished before 'Y' begins.



(B_1 is dummy)

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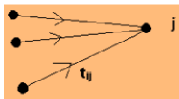
Earliest Start & Finish Times

- Earliest Start Time
  - Earliest time an activity can start
  - For an activity (i, j) it is denoted by ES_i
- Earliest Finish Time
  - Earliest time an activity can finish
  - For an activity (i, j) it is denoted by ES_j



- For the activity (i, j), $ES_j = ES_i + t_{ij}$
- If more than one activity end at the node (j) then

$ES_j = \max\{ES_i + t_{ij}\}$, for all i

For the initial activity $ES_1 = 0$

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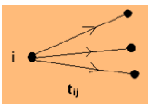
Latest Start & Finish Times

Starting from the last node, we proceed backward to calculate the latest times.



- Latest Start Time
  - Latest time at which an activity can start
  - For an activity (i, j) it is denoted by LS_i
- Latest Finish Time
  - Latest time by which an activity can be completed
  - For an activity (i, j) it is denoted by LS_j
- For the activity (i, j), $LS_i = LS_j - t_{ij}$
- If more than one activity start from the node (i) then

$LS_i = \min\{LS_j - t_{ij}\}$, for all j

For the last node $ES_n = LS_n$

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Total, Free, and Independent Float

- Total Float : Total float of an activity (i, j) is the difference between the maximum time available to finish the activity and the time required to complete it. It is calculated as:

  Total Float = $LS_j - ES_i - t_{ij}$

- Free Float : Free float is the time by which an activity can be delayed beyond its earliest finish without affecting the earliest start time of a succeeding activity. Therefore,

  Free Float = Total Float – ($LS_j - ES_j$)

- Independent Float : The time by which an activity can be rescheduled without affecting the preceding or the succeeding activities is known as independent float.

  Independent Float = Total Float – ($LS_i - ES_i$)

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Critical Path

- Critical Activity: An activity is said to be critical if a delay in its start will cause a further delay in the completion of the entire project.

  Alternate Definition: An activity is said to be critical if its total float is zero.
- Critical Path: The path formed by all the critical activities is called the critical path.
- For the activity (i, j) to lie on the critical path, the following conditions must satisfy:

1. $ES_i = LS_i$
2. $ES_j = LS_j$
3. $ES_j - ES_i = LS_j - LS_i = t_{ij}$

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Procedure to Find Critical Path

1. Draw the network diagram of the project and indicate the time for each activity.
2. Calculate the earliest starting time at each node.
3. Calculate the latest starting time at each node.
4. Prepare a table giving the time t_{ij}, ES_i, ES_j, LS_i, and LS_j for each activity.
5. Find the difference between the earliest time ES_i and the latest time LS_j. Note down the difference (float) $LS_i - ES_i$, $(LF_j - EF_j)$ for each activity.
6. Choose the critical activities with zero float and form the critical path by double line.
7. The sum Σt_{ij} of the critical activities gives the time of completion of the project. (ES_n also denotes the time of completion of the project.)

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Example 1

A project consists of a series of jobs $A, B, C, D, E, F, G, H, I$ such that $A < D, E; B, D < F; C < G; B < H; F, G < I$

The time of completion of each job is given below:

Job	A	B	C	D	E	F	G	H	I
Time (days)	23	8	20	16	24	18	19	4	10

Find the critical path and the minimum time required to complete the project.

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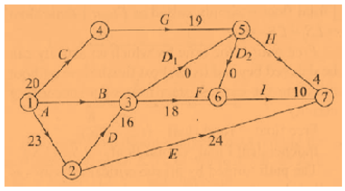
Solution (Ex 1)



D_1 and D_2 are dummy activities with $t_{ij} = 0$

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Earliest Start Time Calculation (Ex 1)

$$
\begin{aligned}
ES_1 &= 0 \quad ES_2 = ES_1 + t_{12} = 0+23 = 23\\
ES_3 &= \text{Max}\,[ES_1 + t_{13}, ES_2 + t_{23}]\\
&= \text{Max}\,[0 + 8, 23 + 16] = 39\\
ES_4 &= ES_1 + t_{14} = 0 + 20 = 20\\
ES_5 &= \text{Max}\,[ES_4 + t_{45}, ES_3 + t_{35}]\\
&= \text{Max}\,[20 + 19, 39 + 0] = 39\\
ES_6 &= \text{Max}\,[ES_3 + t_{35}, ES_5 + T_{56}]\\
&= \text{Max}\,[39 + 18, 39 + 0] = 57\\
ES_7 &= \text{Max}\,[ES_2 + t_{27}, ES_6 + T_{67}, ES_5 - t_{57}]\\
&= \text{Max}\,[23 - 24, 57 + 10, 39 + 4] = 67
\end{aligned}
$$

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Latest Start Time Calculation (Ex 1)

$$
\begin{aligned}
LS_7 &= ES_7 = 67\\
LS_6 &= LS_7 - t_{67} = 67 - 10 = 57\\
LS_5 &= \text{Min}\,[LS_6 \quad t_{56}, LS_7 - t_{57}]\\
&= \text{Min}\,[57 - 0, 67 - 4] = 57\\
LS_4 &= LS_5 - t_{45} = 57 - 19 = 38\\
LS_3 &= \text{Min}\,[LS_6 - t_{36}, LS_5 - t_{35}]\\
&= \text{Min}\,[57 - 18, 57 - 0] = 39\\
LS_2 &= \text{Min}\,[LS_3 - t_{23}, LS_7 - t_{27}]\\
&= \text{Min}\,[39 - 16, 67 - 24] = 23\\
LS_1 &= \text{Min}\,[LS_4 - t_{14}, LS_3 - t_{13}, LS_2 - t_{12}]\\
&= \text{Min}\,[38 - 20, 39 - 8, 23 - 23] = 0
\end{aligned}
$$

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Float & Critical Path Calculation (Ex 1)

Job	Time	Earliest time		Latest time		Float
		Start ES_i	Finish ES_j	Start LS_i	Finish LS_j	
(1, 2)	23	0	23	0	23	0
(1, 3)	8	0	8	31	39	31
(1, 4)	20	0	20	18	38	18
(2, 3)	16	23	39	23	39	0
(2, 7)	24	23	47	43	67	20
(3, 5)	0	39	39	57	57	18
(3, 6)	18	39	57	39	57	0
(4, 5)	19	20	39	38	57	18
(5, 6)	0	39	39	57	57	18
(5, 7)	4	39	43	63	67	24
(6, 7)	10	57	67	57	67	0

Critical activities:
(1, 2), (2, 3), (3, 6), (6, 7)

Critical Path:
1-2-3-6-7

Project Completion time:
23+16+18+10 = 67 days

$[EF_j = ES_i + t_{ij}; LS_i = LF_j - t_{ij} = LS_j - t_{ij}$
Float $= LS_i - ES_i = LS_j - ES_j]$

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Example 2

A project consists of jobs A, B, C, D, E, F, G, H, I such that $A < D$; $A < E$; $B < F$; $D < F$; $C < G$; $C < H$; $F < I$; $G < I$

The time taken for each job is given below:

Job	A	B	C	D	E	F	G	H	I
Time (days)	8	10	8	10	16	17	18	14	9

Draw the network diagram. Find the critical path and minimum time of completion of the project.

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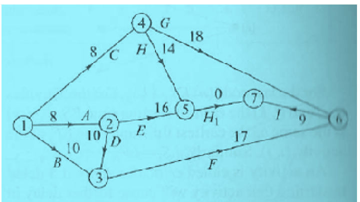
Solution (Ex 2)



(H_1 is a dummy activity)

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Earliest Start Times (Ex 2)

$ES_1 = 0 \quad ES_2 = ES_1 + t_{12} = 0 + 8 = 8$
$ES_3 = \text{Max}\,[ES_1 + t_{13}, ES_2 + t_{23}]$
$= \text{Max}\,[0 + 10, 8 + 10] = 18$
$ES_4 = ES_1 + t_{14} = 0 + 8 = 8$
$ES_5 = \text{Max}\,[ES_2 + t_{25}, ES_4 + t_{45}]$
$= \text{Max}\,[8 + 16, 8 + 14] = 24$
$ES_6 = \text{Max}\,[ES_4 + t_{46}, ES_3 + t_{36}]$
$= \text{Max}\,[8 + 18, 18 + 17] = 35$
$ES_7 = \text{Max}\,[ES_5 + t_{57}, ES_6 + t_{67}]$
$= \text{Max}\,[24 + 0, 35 + 9] = 44$
$ES_7 = ES_7 = 44$

$LS_6 = LS_7 - t_{67} = 44 - 9 = 35$
$LS_5 = LS_7 - t_{57} = 44 - 0 = 44$
$LS_4 = \text{Min}\,[LS_6 - t_{46}, LS_5 - t_{45}]$
$= \text{Min}\,[35 - 18, 44 - 14] = 17$
$LS_3 = LS_5 - t_{36} = 35 - 17 = 18$
$LS_2 = \text{Min}\,[LS_5 - t_{25}, LS_3 - t_{23}]$
$= \text{Min}\,[44 - 16, 18 - 10] = 8$
$LS_1 = \text{Min}\,[LS_4 - t_{14}, LS_3 - t_{13}, LS_2 - t_{12}]$
$= \text{Min}\,[17 - 8, 18 - 10, 8 - 8] = 8$

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Float & Critical Path Calculation (Ex 2)

Job	Time	Earliest time		Latest time		Float	
		Start ES_i	Finish ES_j	Start LS_i	Finish LS_j		
(1, 2)	8	0	8	0	8	0	→
(1, 3)	10	0	10	8	18	8	
(1, 4)	8	0	8	9	17	9	
(2, 3)	10	8	18	8	18	0	→
(2, 5)	16	8	24	28	44	20	
(3, 6)	17	18	35	18	35	0	→
(4, 6)	18	8	26	17	35	9	
(4, 5)	14	8	22	30	44	22	
(5, 7)	0	24	24	44	44	20	
(6, 7)	9	35	44	35	44	0	→

Critical Path:
1-2-3-6-7

Project Completion time = 44 units

$[EF_j = ES_i + t_{ij}; LS_i = LF_j - t_{ij} = LS_j - t_{ij}$
Float $= LS_i - ES_i = LS_j - ES_j]$

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Project Evaluation and Review Technique (PERT)

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Project Planning

- Resource Availability and/or Limits
  - Due date, late penalties, early completion incentives
  - Budget
- Activity Information
  - Identify all required activities
  - Estimate the resources required (time) to complete each activity
  - Immediate predecessor(s) to each activity needed to create interrelationships

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PERT

- PERT is based on the assumption that an activity's duration follows a probability distribution instead of being a single value
- Three time estimates are required to compute the parameters of an activity's duration distribution:
  - pessimistic time (t_p) - the time the activity would take if things did not go well
  - most likely time (t_m) - the consensus best estimate of the activity's duration (assumed to follow Beta distribution)
  - optimistic time (t_o) - the time the activity would take if things did go well

Mean (expected time): $t_e = \frac{t_p + 4\,t_m + t_o}{6}$

Variance: $V_t = \sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$

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PERT Analysis

- Draw the network.
- Analyze the paths through the network and find the critical path.
- The length of the critical path is the mean of the project duration probability distribution which is assumed to be normal.
- The standard deviation of the project duration probability distribution is computed by adding the variances of the critical activities (all of the activities that make up the critical path) and taking the square root of that sum.
- Probability computations can now be made using the normal distribution table.

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Probability Computation

Determine probability that project is completed within specified time

$$Z = \frac{x - \mu}{\sigma}$$

where $\mu = t_p$ = project mean time

σ = project standard mean time

x = (proposed) specified time

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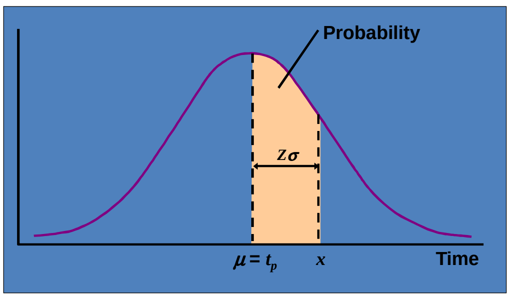
Normal Distribution of Project Time



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PERT Algorithm

- Step 1: List all activities and the draw a network diagram.
- Step 2: Denote the most likely time by t_m, optimistic time by t_o, pessimistic time by t_p, and compute the expected time as:

  $t_e = (t_m + t_o + t_p)/6$
- Step 3: Tabulate expected activity times, ES time, LF time for each event.
- Step 4: Determine the total float (ES - LF) for each activity.
- Step 5: Identify the critical path & calculate the project duration.
- Step 6: Compute variance of each activity using t_p, and t_o.
- Step 7: Calculate the standard normal variate :

  z_0 = (Due date – Expected completion date)/Project s.d.
- Step 8: Use standard normal table to compute the probability of completing the project within the scheduled time.

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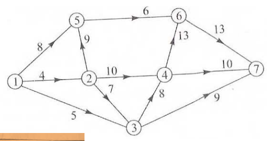
Example 6

- Find the critical path of the project given by the following network diagram:



Job	(1, 2)	(1, 3)	(1, 5)	(2, 3)	(2, 4)	(2, 5)	(3, 4)	(3, 7)	(4, 6)	(4, 7)	(5, 6)	(6, 7)
t_o	2	2	5	6	9	5	4	6	12	7	3	10
t_m	4	5	8	7	10	9	8	9	13	10	6	13
t_p	6	8	11	8	11	13	12	12	14	13	9	16

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Solution (EX 6)

Job	t_o	t_m	t_p	t_e
(1, 2)	2	4	6	4
(1, 3)	2	5	8	5
(1, 5)	5	8	11	8
(2, 3)	6	7	8	7
(2, 4)	9	10	11	10
(2, 5)	5	9	13	9
(3, 4)	4	8	12	8
(3, 7)	6	9	12	9
(4, 6)	12	13	14	13
(4, 7)	7	10	13	10
(5, 6)	3	6	9	6
(6, 7)	10	13	16	13

$E(\mu_1) = 0$; $E(\mu_2) = 4$

$E(\mu_3) = \text{Max}\ [4 + 7, 0 + 5] = 11$

$E(\mu_4) = \text{Max}\ [4 + 10, 11 + 8] = 19$

$E(\mu_5) = \text{Max}\ [0 + 8, 4 + 9] = 13$

$E(\mu_6) = \text{Max}\ [19 + 13, 13 + 6] = 32$

$E(\mu_7) = \text{Max}\ [19 + 10, 11 + 9, 32 + 13] = 45$

$E(L_7) = 45$; $E(L_6) = 32$; $E(L_5) = 26$

$E(L_4) = \text{Min}\ [45 - 10, 32 - 13] = 19$

$E(L_3) = \text{Min}\ [45 - 9, 19 - 8] = 11$

$E(L_2) = \text{Min}\ [11 - 7, 19 - 10, 26 - 6] = 4$

$E(L_1) = \text{Min}\ [4 - 4, 11 - 5, 26 - 8] = 0$

Event	$E(\mu_i)$	$E(L_i)$	Slack	
1	0	0	0	→
2	4	4	0	→
3	11	11	0	→
4	19	19	0	→
5	13	26	13	
6	32	32	0	→
7	45	45	0	→

The critical path is 1 – 2 – 3 – 4 – 6 – 7

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Example 7

A project consists of the following activities and time estimates.

Activity	Estimated duration in weeks		
	Optimistic	Most likely	Pessimistic
(1, 2)	1	1	7
(1, 3)	1	4	7
(1, 4)	2	2	8
(2, 5)	1	1	1
(3, 5)	2	5	14
(4, 6)	2	5	8
(5, 6)	3	6	15

(i) Draw the network.

(ii) Find the expected time and variance for each activity.

(iii) What is the probability that the project will be completed 4 weeks earlier than the expected time.

(iv) What is the probability that the project will be completed in 19 weeks.

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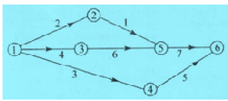
Solution (EX 7)



$E(\mu_1) = 0$; $E(\mu_2) = 2$; $E(\mu_3) = 4$

$E(\mu_4) = 3$; $E(\mu_5) = \text{Max}\ (2 + 1, 4 + 6) = 10$

$E(\mu_6) = \text{Max}\ \{10 + 7, 3 + 5\} = 17$

$E(L_6) = 17$; $E(L_5) = 10$; $E(L_4) = 12$

$E(L_3) = 4$; $E(L_2) = 9$;

$E(L_1) = \text{Min}\ \{9 - 2, 4 - 4, 12 - 3\} = 0$

Activity	t_o	t_m	t_p	t_e	σ^2
(1, 2)	1	1	7	2	1
(1, 3)	1	4	7	4	1
(1, 4)	2	2	8	3	1
(2, 5)	1	1	1	1	0
(3, 5)	2	5	14	6	4
(4, 6)	2	5	8	5	1
(5, 6)	3	6	15	7	4

Event	$E(\mu_i)$	$E(L_i)$	Slack	
1	0	0	0	→
2	2	9	7	
3	4	4	0	→
4	3	12	9	
5	10	10	0	→
6	17	17	0	→

The critical events are 1, 3, 5, 6. Hence the critical path is 1 – 3 – 5 – 6.

Expected time of completion of the project is 17 weeks $\therefore T = 17$

$\sigma^2 = \sigma_{1,3}^2 + \sigma_{3,5}^2 + \sigma_{5,6}^2 = 1 + 4 + 4 = 9$

$\therefore \sigma = 3$

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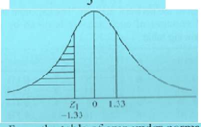
Solution (EX 7)

(iii) To find the probability that the project would be completed 4 weeks earlier:

We have $Z = \frac{t - 17}{3}$

Given $t = 17 - 4 = 13$

$\therefore \quad Z_1 = \frac{13 - 17}{3} = -1.33$

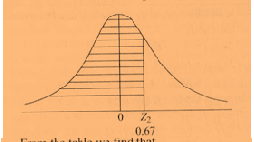


From the table of area under normal curve we have $P[Z \le 1.33] = 0.91 = 0.5 + 0.41$

$\therefore \quad P[Z \le -1.33] = 0.5 - 0.41 = 0.09$

(iv) To find the probability that the project would be completed in 19 weeks:

We have $Z_2 = \frac{19 - 17}{3} = 0.67$



From the table we find that

$\therefore P[Z \le 0.67] = 0.75$

Thus the project would be completed 4 weeks earlier with a probability of 0.09 and in 19 weeks with a probability of 0.75.

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Benefits of CPM/PERT

- Useful at many stages of project management
- Mathematically simple
- Give critical path and slack time
- Provide project documentation
- Useful in monitoring costs

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Continued…

CPM/PERT can answer the following important questions:

- How long will the entire project take to be completed? What are the risks involved?
- Which are the critical activities or tasks in the project which could delay the entire project if they were not completed on time?
- Is the project on schedule, behind schedule or ahead of schedule?
- If the project has to be finished earlier than planned, what is the best way to do this at the least cost?

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Limitations of CPM/PERT

- Clearly defined, independent and stable activities
- Specified precedence relationships
- Over emphasis on critical paths
- Deterministic CPM model
- Activity time estimates are subjective and depend on judgment
- PERT assumes a beta distribution for these time estimates, but the actual distribution may be different
- PERT consistently underestimates the expected project completion time due to alternate paths becoming critical

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Computer Software for Project Management

- *Microsoft Project* (Microsoft Corp.)
- *MacProject* (Claris Corp.)
- *PowerProject* (ASTA Development Inc.)
- *Primavera Project Planner* (Primavera)
- *Project Scheduler* (Scitor Corp.)
- *Project Workbench* (ABT Corp.)

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