Linear Programming

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Lecture Outline

- Introduction to Linear Programming (LP)
- Historical Perspective
- Model Formulation
- Graphical Solution Method
- Simplex Method

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Introduction to LP

- Today many of the resources needed as inputs to operations are in limited supply.
- Operations managers must understand the impact of this situation on meeting their objectives.
- Linear programming (LP) is one way that operations managers can determine how best to allocate their scarce resources.

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Continued…

Linear Programming is a mathematical modeling technique used to find optimal solutions to complex problems. It is used to determine the best ways of allocating their resources to minimize costs or to maximize profits.

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Historical Perspective

- 1928 – John von Neumann published related central theorem of game theory
- 1944 – Von Neumann and Morgenstern published Theory of Games and Economic Behavior
- 1936 – W.W. Leontief published "Quantitative Input and Output Relations in the Economic Systems of the US" which was a linear model without objective function.
- 1939 – Kantoravich (Russia) actually formulated and solved a LP problem
- 1941 – Hitchcock poses transportation problem (special LP)
- WWII – Allied forces formulate and solve several LP problems related to military

A breakthrough occurred in 1947...

Types of Linear Programming

There are five common types of decisions in which LP may play a role

- Product mix
- Production planning
- Blending problem (ingredient mix)
- Transportation
- Assignment (flow capacity)

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Product Mix Problem

- A manufacturer has fixed amounts of different resources such as raw material, labor, and equipment.
- These resources can be combined to produce any one of several different products.
- The quantity of the i^{th} resource required to produce one unit of the j^{th} product is known.
- The decision maker wishes to produce the combination of products that will maximize total income.

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Production Planning Problem

- A manufacturer knows that he must supply a given number of items of a certain product each month for the next *n* months.
- They can be produced either in regular time, subject to a maximum each month, or in overtime. The cost of producing an item during overtime is greater than during regular time. A storage cost is associated with each item not sold at the end of the month.
- The problem is to determine the production schedule that minimizes the sum of production and storage costs.

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Blending Problem (ingredient mix)

- Blending problems refer to situations in which a number of components (or commodities) are mixed together to yield one or more products.
- Typically, different commodities are to be purchased. Each commodity has known characteristics and costs.
- The problem is to determine how much of each commodity should be purchased and blended with the rest so that the characteristics of the mixture lie within specified bounds and the total cost is minimized.

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Transportation Problem

- A product is to be shipped in the amounts $a_1, a_2, \ldots, a_m$ from *m* shipping origins and received in amounts $b_1, b_2, \ldots, b_n$ at each of *n* shipping destinations.
- The cost of shipping an unit from the i^{th} origin to the j^{th} destination is known.
- The problem is to determine the amount to be shipped from each origin to each destination such that the total transportation cost is a minimum.

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Assignment (flow capacity) Problem

- One or more commodities (e.g., traffic, water, information, cash, etc.) are flowing from one point to another through a network whose branches have various constraints and flow capacities.
- The direction of flow in each branch and the capacity of each branch are known.
- The problem is to determine the maximum flow, or capacity of the network.

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Characteristics of LP Problems

- A well-defined single objective must be stated.
- The total achievement of the objective must be constrained by limited resources.
- The objective and each of the constraints must be expressed as linear mathematical functions.

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LP Model Formulation

- Decision variables
  - mathematical symbols representing levels of activity of an operation
- Objective function
  - a linear relationship reflecting the objective of an operation
  - most frequent objective of business firms is to *maximize profit*
  - most frequent objective of individual operational units (such as a production or packaging department) is to *minimize cost*
- Constraint
  - a linear relationship representing a restriction on decision making

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Steps in Formulating LP Problems

1. Define the decision variables (positive)
2. Define the objective type (min or max)
3. Write the mathematical function for the objective
4. Write a 1- or 2-word description of each constraint
5. Write the constraints using mathematical function
6. Rewrite the problem in final form: Defined objective subject to all constraints and non-negativity restrictions

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LP Model Formulation

Max/min $z = c_1x_1 + c_2x_2 + \dots + c_nx_n$

subject to:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \ (\le, =, \ge)\ b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \ (\le, =, \ge)\ b_2 \\ \vdots \\ a_{m1}x1 + a_{m2}x_2 + \dots + a_{mn}x_n \ (\le, =, \ge)\ b_m \end{cases}$$

x_j = decision variables
b_i = constraint levels
c_j = objective function coefficients
a_{ij} = constraint coefficients

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LP Formulation: Example 1

Cycle Trends is introducing two new lightweight bicycle frames, the Deluxe and the Professional, to be made from aluminum and steel alloys. The anticipated unit profits are $10 for the Deluxe and $15 for the Professional.

The number of pounds of each alloy needed per frame is summarized on the next slide. A supplier delivers 100 pounds of the aluminum alloy and 80 pounds of the steel alloy weekly. How many Deluxe and Professional frames should Cycle Trends produce each week?

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LP Formulation: Example 1

Pounds of each alloy needed per frame

	Aluminum Alloy	Steel Alloy
Deluxe	2	3
Professional	4	2

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LP Formulation: Example 1

Define the decision variables
- x_1 = number of Deluxe frames produced weekly
- x_2 = number of Professional frames produced weekly

Define the objective
- Maximize total weekly profit

Write the mathematical objective function
- Max $Z = 10x_1 + 15x_2$

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LP Formulation: Example 1

Write a one- or two-word description of each constraint

- Aluminum available
- Steel available

Write the constraints in mathematical form

- $2x_1 + 4x_2 \leq 100$
- $3x_1 + 2x_2 \leq 80$

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LP Formulation: Example 1

- LP in Final Form

Max $Z = 10x_1 + 15x_2$

subject to

$2x_1 + 4x_2 \leq 100$ (aluminum constraint)

$3x_1 + 2x_2 \leq 80$ (steel constraint)

$x_1 , x_2 \geq 0$

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LP Formulation: Example 2

Montana Wood Products manufacturers two-high quality products, tables and chairs. Its profit is $15 per chair and $21 per table. Weekly production is constrained by available labor and wood. Each chair requires 4 labor hours and 8 board feet of wood while each table requires 3 labor hours and 12 board feet of wood. Available wood is 2400 board feet and available labor is 920 hours. Management also requires at least 40 tables and at least 4 chairs be produced for every table produced. To maximize profits, how many chairs and tables should be produced?

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LP Formulation: Example 2

LP in Final Form

Max $Z = 15x_1 + 21x_2$

Subject to

$4x_1 + 3x_2 \leq 920$ (labor constraint)

$8x_1 + 12x_2 \leq 2400$ (wood constraint)

$x_2 - 40 \geq 0$ (make at least 40 tables)

$x_1 - 4x_2 \geq 0$ (at least 4 chairs for every table)

$x_1 , x_2 \geq 0$ (non-negativity constraints)

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Important Definitions

- Basic Solution: Consider a set of m linear simultaneous equations of n $(n > m)$ variables:

  $Ax = b.$

  Let, $r(A) = r(A, b) = m.$

  If any $m \times m$ non-singular matrix be chosen from A and if the remaining $(n – m)$ variables not associated with the chosen matrix is set to zero then the solution of the resulting system of equations is a basic solution.

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Important Definitions

- Non-degenerate Basic Solution

  In a basic solution, if none of the m basic variables are zero then the solution is called non-degenerate basic solution.

- Degenerate Basic Solution

  In a basic solution, if at least one basic variable is zero then the solution is called degenerate basic solution.

  - **Non-degenerate basic solution has exactly *m* non-zero & *(n-m)* zero variables.**
  - **Degenerate basic solution has more than *(n-m)* zero variables.**
  - **Number of basic solutions: ^nC_m**

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Important Definitions

- Feasible Solution

  In a LPP, a solution satisfying the constraints & non-negativity conditions is called feasible solution.
- Basic Feasible Solution

  A feasible solution which is basic is called a basic feasible solution.
- Non-degenerate Basic Feasible Solution

  A basic feasible solution in which no basic variable is zero is called a non-degenerate basic feasible solution.

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Important Definitions

- Degenerate Basic Feasible Solution

  A basic feasible solution in which at least one basic variable is zero is called a degenerate basic feasible solution.
- Optimum Solution

  A feasible solution which optimizes the objective function of a LPP is called an optimum solution.

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Nature of Solution of a LPP

- Unique solution
- Infinite number of optimum solutions
- Unbounded solution
- No feasible solution

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Graphical Solution Method

1. Plot model constraint on a set of coordinates in a plane
2. Identify the feasible solution space (region) on the graph where all constraints are satisfied simultaneously
3. Calculate the objective value at all the corner points of the feasible region & select the optimum (max or min) point (solution)

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Example: LP Formulation

	RESOURCE REQUIREMENTS		
PRODUCT	*Labor (hr/unit)*	*Clay (lb/unit)*	*Revenue ($/unit)*
Bowl	1	4	40
Mug	2	3	50

There are 40 hours of labor and 120 pounds of clay available each day. Formulate the LPP.

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Example: LP Formulation

Decision variables

x_1 = number of bowls to produce

x_2 = number of mugs to produce

Maximize $Z = \$40\, x_1 + \$50\, x_2$

Subject to

$x_1 + 2x_2 \leq 40$ hr (labor constraint)

$4x_1 + 3x_2 \leq 120$ lb (clay constraint)

$x_1, x_2 \geq 0$

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Graphical Solution: Example 1

Maximize $Z = \$40\,x_1 + \$50\,x_2$

Subject to

$$\begin{aligned} x_1 + 2x_2 &\le 40 \text{ hr} \quad \text{(labor constraint)} \\ 4x_1 + 3x_2 &\le 120 \text{ lb} \quad \text{(clay constraint)} \\ x_1, x_2 &\ge 0 \end{aligned}$$

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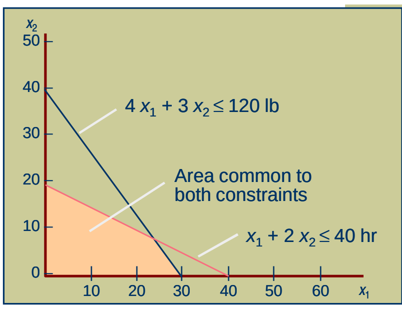
Graphical Solution: Example 1



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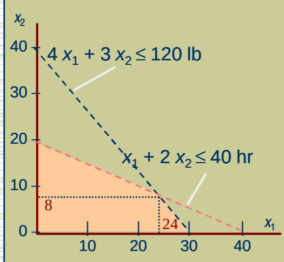
Graphical Solution: Example 1



$$\begin{aligned} x_1 + 2x_2 &= 40 \\ 4x_1 + 3x_2 &= 120 \\ \hline 4x_1 + 8x_2 &= 160 \\ -4x_1 - 3x_2 &= -120 \\ \hline 5x_2 &= 40 \\ x_2 &= 8 \\ \\ x_1 + 2(8) &= 40 \\ x_1 &= 24 \end{aligned}$$

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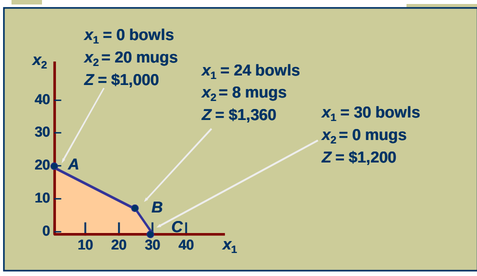
Graphical Solution: Example 1



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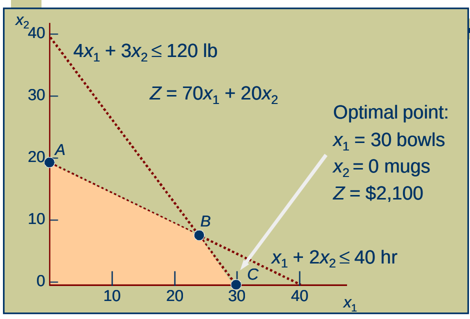
Graphical Solution: Example 1



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Graphical Solution: Example 2

Minimize $Z = x_1 + x_2$

Subject to

$$\begin{aligned} 5x_1 + 9x_2 &\le 45 \\ x_1 + x_2 &\ge 2 \\ x_2 &\le 4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

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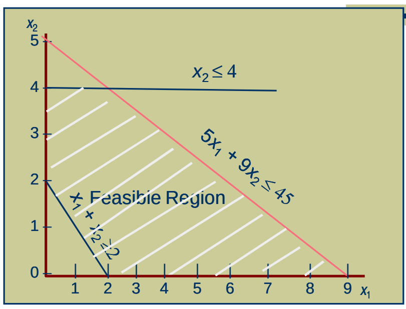
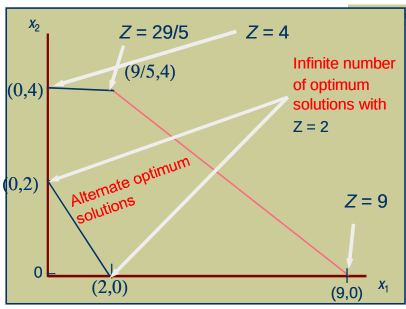
Graphical Solution: Example 2



Graphical Solution: Example 2



Graphical Solution: Example 3

Maximize $Z = 2x_1 - x_2$

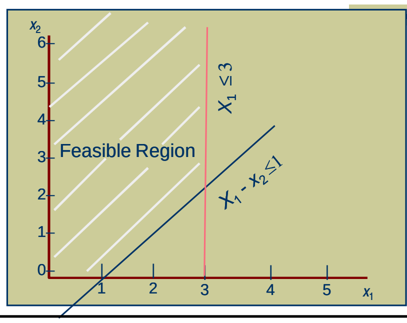
Subject to

$x_1 - x_2 \le 1$

$x_1 \le 3$

$x_1, x_2 \ge 0$

Graphical Solution: Example 3



Graphical Solution: Example 3