Game Theory

RK Jana

1

An Example

Two players A & B are playing a game.

Game: A & B will simultaneously & independently write one of the numbers from 1, 2 & 3.

Rule: If the sum of numbers is even then A wins & B pays the amount to A. If the sum of numbers is odd then B wins & A pays the amount to A.

		B		
		1	2	3
A	1	2	-3	4
	2	-3	4	-5
	3	4	-5	6

2

History of Game Theory

- Originated by von Neumann in 1928
- von Neumann and Morganstern, 1944
  - Zero-sum games
- John Nash
  - Nonzero-sum games
- Harsanyi, Selten
  - Incomplete information

Developed to explain the optimal strategy in two-person interactions.

3

More on Game Theory

- Game theory is the general theory of strategic behavior.
  - Generally depicted in mathematical form
  - Plays an important role in modern economics
- Other application areas: computer science, biology, psychology, politics, sociology, etc.
- Whenever people or other agents interact with each other they are playing a game
- Classical example: Chess, Draughts

4

Rules & Payoffs

- The rules of the game state who can do what, and when they can do it.
- A player's payoff is the amount that the player wins or loses in a particular situation in a game. The matrix associate with the game indicates the payoffs and is known as the payoff matrix.

5

Strategy

- A player's strategy is a course of actions in each possible situation of the game.
- Pure Strategy: Pure strategy is a decision in making rule in which one particular course of action is selected.
- Mixed Strategy: Mixed strategy is a decision making rule in which a player selects his course of action from all the pure strategies with some definite probability.
- Optimal Strategy: It is the course of action in which the player optimizes the possible loss or gain.
- A players has a dominant strategy if that player's best strategy does not depend on what other players do.

6

Nash Equilibrium

Nash equilibrium occurs when each player's strategy is optimal, given the strategies of the other players.

- A player's best strategy is the strategy that maximizes that player's payoff, given the strategies of other players.
- A Nash equilibrium is a situation in which each player makes his or her best response.

7

Classification of Games

- Two-person Game
- N-person Game
- Zero Sum Game
- Two-person-zero-sum-game
- Finite Game
- Infinite Game

8

Prisoner's Dilemma: A Famous Example of Game Theory

- Two people X & Y are arrested and charged with a serious crime. They are kept in separate cells, with no opportunity to confer.

		Y: Confess	Y: Not Confess
X	Confess	4 years each	1 year for X and 8 years for Y
	Not Confess	8 years for X and 1 year for Y	3 years each

9

Continued…

- Strategies must be undertaken without the full knowledge of what other players will do.
- Players adopt dominant strategies, but they don't necessarily lead to the best outcome.

10

The Maximin & Minimax Criterion

		B: I	B: II
A	I	-2	2
	II	-1	4
	III	2	3

The minimum guaranteed gain of the maximizing player
= Maximizing minimum loss of the minimizing player
= Value of the game

11

Continued…

If a player lists the worst possible outcomes of all his potential strategies, he will choose the strategy to be the most suitable one which corresponds to the best of these worst outcomes.

12

Saddle Point

If in a game the maximin for player A is equal to the minimax for player B then the game is said to have a saddle point.

Saddle point is the position in the game matrix where maximum of the row minimum (maximin) coincides with the minimum of the column maximum (minimax).

Maximum of the row minimum = Minimum of the column maximum = Value of the game

Solving a Game with a Saddle Point: Example 1

Solve the following game:

B

A	1	2	3	4
I	-5	3	1	20
II	5	5	4	6
III	-4	-2	0	-5

Continued…

$\boxed{-5}$	3	1	20
5	5	$\boxed{4}$	6
-4	-2	0	$\boxed{-5}$

Row minimum = Column maximum = Value of the game

Example 2

For what value of "a", the game is determinable?

B

A	I	II	III
I	a	5	2
II	-1	a	-8
III	-2	3	a

Continued…

Ignoring "a", we can find:
maximum of row minimum (maximin) = 2
minimum of column maximum (minimax) = -1

The game will be determinable with value *v* if
$-1 \le v \le 2$
So, $-1 \le a \le 2$.

Example 3

Solve the following game:

B

A	B1	B2	B3	B4	B5
A1	4	2	1	7	-1
A2	1	-4	-6	-7	6
A3	3	2	3	4	2
A4	-6	1	-1	0	4
A5	0	0	6	0	0

Value of the Game = 2

Example 4

For the following game, determine the number of saddle points and the value of the game.

B

		I	II	III
A	I	1	2	1
	II	0	-4	-1
	III	1	3	-2

 # saddle points = 2
(I, I), (I, III)
value of the game = 1

Games without a Saddle Point

- Maximin-minimax criteria do not satisfy.
- Players do not have a single best strategy.
- Each player can improve his payoff by selecting a different strategy.
- Players have to adopt mixed strategy.
- The game becomes unstable.

Solution of 2×2 Game

B

		B1	B2
A	A1	a_{11}	a_{12}
	A2	a_{21}	a_{22}

x_1 : probability of choosing strategy A1 by A
x_2 : probability of choosing strategy A2 by A
y_1 : probability of choosing strategy B1 by B
y_2 : probability of choosing strategy B2 by B
v : value of the game

$x_1 + x_2 = 1$
$y_1 + y_2 = 1$

Continued…

$$x_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})} \qquad y_1 = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})}$$

$$x_2 = \frac{a_{11} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})} \qquad y_2 = \frac{a_{11} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})}$$

$$v = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})}$$

Example 5

Solve the following 2×2 game:

	B1	B2
A1	-2	5
A2	7	-6

Continued…

$$x_1 = \frac{13}{20} \qquad y_1 = \frac{11}{20} \qquad v = \frac{23}{20}$$

$$x_2 = \frac{7}{20} \qquad y_2 = \frac{9}{20}$$

The Method of Dominance

- If all the elements of the i-th row be less than or equal to the corresponding elements of any other row, say r-th, then the r-th row dominates the i-th row and we discard it.
- If all the elements of the j-th column be greater than or equal to the corresponding elements of any other column, say p-th column, then the p-th column dominates the j-th column and we discard it.
- If the i-th row be dominated by a convex combination of other rows then the i-th is deleted from the payoff marix. Similarly, if the j-th column dominates a convex combination of other columns then the j-th column is deleted from the payoff matrix.

Example 6

- Use dominance to solve the following game:

	B1	B2	B3	B4
A1	3	2	4	0
A2	3	4	2	4
A3	4	2	4	0
A4	0	4	0	8

Continued…

All the elements of A1 are less than or equal to the corresponding elements of A3. So, A3 dominates A1 & hence A1 is deleted.

	B1	B2	B3	B4
A2	3	4	2	4
A3	4	2	4	0
A4	0	4	0	8

Continued…

All the elements of B3 are less than or equal to the corresponding elements of B1. So, B3 dominates B1 & hence B1 is deleted.

	B2	B3	B4
A2	4	2	4
A3	2	4	0
A4	4	0	8

Continued…

The convex combination of B3 & B4 (B3 + B4)/2 dominates B2. So, B2 is deleted.

	B3	B4
A2	2	4
A3	4	0
A4	0	8

Continued…

Again, the convex combination of A3 & A4 (A3 + A4)/2 dominates A2. So, A2 is deleted.

	B3	B4
A3	4	0
A4	0	8

Continued…

$$x_3 = \frac{2}{3} \qquad y_3 = \frac{2}{3} \qquad v = \frac{8}{3}$$

$$x_4 = \frac{1}{3} \qquad y_4 = \frac{1}{3}$$

Final Solution:
A (0, 0, 2/3, 1/3)
B (0, 0, 2/3, 1/3)
v = 8/3

31

Home Work: 1

Use dominance to solve the game:

	B1	B2	B3	B4
A1	8	15	-4	-2
A2	19	15	17	16
A3	0	20	15	5

Solution:
A (0, 15/16, 1/16)
B (0, 11/16, 0, 5/16)
v = 245/16

32

Home Work: 2

Use dominance to solve the game:

	B1	B2	B3	B4	B5	B6
A1	0	0	0	0	0	0
A2	4	2	0	2	1	1
A3	4	3	1	3	2	2
A4	4	3	7	-5	1	2
A5	4	3	4	-1	2	2
A6	4	3	3	-2	2	2

Solution:
A (0, 0, 6/7, 1/7, 0, 0)
B (0, 0, 4/7, 3/7, 0, 0)
v = 13/7

33

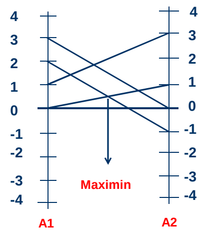
Graphical Method: Solution of 2×n Game

B

A		B1	B2	B3	B4
	A1	1	3	0	2
	A2	3	0	1	-1

34

Continued…



	B3	B4
A1	0	2
A2	1	-1

Solution:
A (1/2, 1/2)
B (0, 0, 3/4, 1/4)
v = 1/2

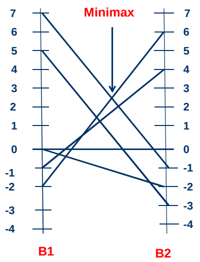
35

Graphical Method: Solution of n×2 Game

	B1	B2
A1	0	-2
A2	7	-1
A3	-1	4
A4	-2	6
A5	5	-3

36

Continued…



	B1	B2
A2	7	-1
A4	-2	6

Solution:
A (0, 1/2, 0, 1/2,0)
B (7/16, 9/16)
v = 5/2

37