Duality

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Lecture Outline

- Key concepts
- Formulation of the dual of LPP
- Duality theorems
- Primal dual relationship
- Duality & simplex method
- Numerical examples

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Introduction

Primal Problem	Dual Problem
Max $Z = c_1x_1 + c_2x_2$ s. t. $a_{11}x_1 + a_{12}x_2 \ge b_1$ $a_{21}x_1 + a_{22}x_2 \ge b_2$ $a_{31}x_1 + a_{32}x_2 \ge b_3$ $a_{41}x_1 + a_{42}x_2 \ge b_4$ $x_1 , x_2 \ge 0$	Min $w = b_1v_1 + b_2v_2 + b_3v_3 + b_4v_4$ s. t. $a_{11}v_1 + a_{21}v_2 + a_{31}v_3 + a_{41}v_4 \le c_1$ $a_{12}v_1 + a_{22}v_2 + a_{32}v_3 + a_{42}v_4 \le c_2$ $v_1 , v_2 , v_3 , v_4 \ge 0$

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Formulation of Dual

- Number of variables in the dual problem is equal to the number of constraints in the primal & vice-versa
- The elements of the requirement vector in one problem are the respective prices in the objective function of the other problem
- If the primal is maximization type then the dual is minimization type & vice-versa
- The “less than equal” sign in primal constraints become “greater than equal” in the dual & vice-versa

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Dual in Vector Notations

The Primal

Max $Z = cx$

s. t. $Ax \le b, x \ge 0$

The Dual

Min $w = b^T v$

s. t. $A^T v \ge c^T, v \ge 0$

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Important Theorems in Duality

- Theorem 1: The dual of the dual is primal.
- Theorem 2: If there exists a feasible solution to both the primal and dual problems such that the objective values are equal then the solutions are optimum to the respective problems.
- Theorem 3: An LPP admits of a finite number of optimum solutions if and only if there exist feasible solutions to both primal & dual problems.

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Continued…

- Theorem 4: If the primal problem has an unbounded objective function then the dual has no feasible solution & vice-versa.
- Theorem 5: If any of the constraints in the primal is a perfect equality, then the corresponding dual variable is unrestricted in sign.
- Theorem 6: If any of the variables in the primal is unrestricted in sign, then the corresponding dual constraint is a perfect equality.

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Useful Summary

Primal Problem	Dual Problem	Conclusion
Feasible solution	Feasible solution	Finite optimum for both
No feasible solution	Feasible solution	Unbounded dual objective
Feasible solution	No feasible solution	Unbounded primal objective
No feasible solution	No feasible solution	No solution exists
Equality constraint	…	Corresponding dual variable unrestricted in sign
Primal variable unrestricted in sign	…	Corresponding dual constraint is equality type

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Regularization of the Primal Problem

Consider the following primal problem:

Maximize

$$Z = 12x_1 + 4x_2$$

subject to:

$$\begin{aligned} 4x_1 + 7x_2 &\le 56 \\ 2x_1 + 5x_2 &\ge 20 \\ 5x_1 + 4x_2 &= 40 \\ x_1 &\ge 0 \\ x_2 &\ge 0 \end{aligned}$$

- The first inequality requires no modification

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Continued…

- The second inequality can be changed to the less-than-or-equal-to type by multiplying both sides of the inequality by -1 and reversing the direction of the inequality; that is,

$$-2x_1 - 5x_2 \le -20$$

- The equality constraint can be replaced by the following two inequality constraints:

$$\begin{aligned} 5x_1 + 4x_2 &\le 40 \\ 5x_1 + 4x_2 &\ge 40 \end{aligned}$$

- If both of these inequality constraints are satisfied, the original equality constraint is also satisfied.

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Continued…

- Multiplying both sides of the inequality by –1 and reversing the direction of the inequality yields:

$$-5x_1 - 4x_2 \le -40$$

- The primal problem can now take the following standard form:

Maximize

$$Z = 12x_1 + 4x_2$$

subject to:

$$\begin{aligned} 4x_1 + 7x_2 &\le 56 \\ -2x_1 - 5x_2 &\le -20 \\ 5x_1 + 4x_2 &\le 40 \\ -5x_1 - 4x_2 &\le -40 \\ x_1 &\ge 0 \\ x_2 &\ge 0 \end{aligned}$$

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Continued…

The dual of this problem can now be obtained as follows:

Min $w = 56v_1 - 20v_2 + 40v_3 - 40v_4$

s. t.

$$\begin{aligned} 4v_1 - 2v_2 + 5v_3 - 5v_4 &\ge 12 \\ 7v_1 - 5v_2 + 4v_3 - 4v_4 &\ge 4 \\ v_1, v_2, v_3, v_4 &\ge 0 \end{aligned}$$

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Example 1

Formulate the dual of the LPP:

Max $Z = 2x_1 + 3x_2 + 4x_3$
s. t.
$x_1 - 5x_2 + 3x_3 = 7$
$2x_1 - 5x_2 + 3x_3 \leq 3$
$3x_2 - x_3 \geq 5$
$x_1, x_2 \geq 0$
and x_3 is unrestricted in sign.

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Example 1

Rewrite the problem by introducing new positive variables: $x_3 = x'_3 - x''_3$

Max $Z = 2x_1 + 3x_2 + 4(x'_3 - x''_3)$
s. t.
$x_1 - 5x_2 + 3(x'_3 - x''_3) \leq 7$
$x_1 - 5x_2 + 3(x'_3 - x''_3) \geq 7$
$2x_1 - 5x_2 + 3(x'_3 - x''_3) \leq 3$
$3x_2 - (x'_3 - x''_3) \geq 5$
$x_1, x_2, x'_3, x''_3 \geq 0$

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Example 1

Writing equivalently:

Max $Z = 2x_1 + 3x_2 + 4(x'_3 - x''_3)$
s. t.
$x_1 - 5x_2 + 3(x'_3 - x''_3) \leq 7$
$-x_1 + 5x_2 - 3(x'_3 - x''_3) \leq -7$
$2x_1 - 5x_2 + 3(x'_3 - x''_3) \leq 3$
$-3x_2 + (x'_3 - x''_3) \leq -5$
$x_1, x_2, x'_3, x''_3 \geq 0$

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Example 1

Min $w = 7v_1 - 7v_2 + 3v_3 - 5v_4$
s. t.
$v_1 - v_2 + 2v_3 \geq 2$
$-5v_1 + 5v_2 - 5v_3 - 3v_4 \geq 3$
$3v_1 - 3v_2 + 3v_3 + v_4 \geq 4$
$-3v_1 + 3v_2 - 3v_3 - v_4 \geq -4$
$v_1, v_2, v_3, v_4 \geq 0$

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Example 1

Final form:

Min $w = 7v_5 + 3v_3 - 5v_4$
s. t.
$v_5 + 2v_3 \geq 2$
$-5v_5 - 5v_3 - 3v_4 \geq 3$
$3v_5 + 3v_3 + v_4 = 4$
$v_3, v_4 \geq 0,$
$v_5 (= v_1 - v_2)$ is unrestricted in sign

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Additional Problems

Find the dual of the following primal problems:

Max $z = x_1 - x_2 + 3x_3 + 2x_4$ s. t. $x_1 + x_2 \geq -1$ $x_1 - 3x_2 - x_3 \leq 7$ $x_1 + x_2 - 3x_4 = -2$ $x_1, x_4 \geq 0$ x_2, x_3 are unrestricted in sign	Min $z = x_3 + x_4 + x_5$ s. t. $x_1 - x_3 + x_4 - x_5 = -2$ $x_2 - x_3 - x_4 + x_5 = 1$ $x_j \geq 0$ for all j

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Duality & Simplex Method

Rule 1

If the primal (dual) variable be related to a slack and/or surplus variable in the dual (primal) problem, its optimum solution is directly read off from the net evaluation row of the optimum dual (primal) simplex table, as the net evaluation corresponding to this slack and/or surplus variable.

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Continued…

Rule 2

If the primal (dual) variable be related to an artificial starting variable in the dual (primal) problem, its optimum value is directly read off from the net evaluation row of the optimum dual (primal) simplex table as the net evaluation relating to this artificial variable after deleting the penalty cost M and changing sign of the net evaluations.

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Continued…

Rule 3

If either problem (primal or dual) has unbounded solution then the other will have no feasible solution.

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Duality: Example 1

Write the dual of the following LPP & hence solve it:

Max $Z = 3x_1 - 2x_2$

s. t.

$x_1 \leq 4$

$x_2 \leq 6$

$x_1 + x_2 \leq 5$

$-x_2 \leq -1$

$x_1, x_2 \geq 0$

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Continued…

The dual

Min $w = 4v_1 + 6v_2 + 5v_3 - v_4$

s. t.

$v_1 + v_3 \geq 3$

$v_2 + v_3 - v_4 \geq -2$

$v_1, v_2, v_3, v_4 \geq 0$

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Continued…

The standard form

Max $w' = -4v_1 - 6v_2 - 5v_3 + v_4 + 0.s_1 + 0.s_2$

s. t.

$v_1 + v_3 - s_1 = 3$

$-v_2 - v_3 + v_4 + s_2 = 2$

$v_j \geq 0$, for $j=1,2,3,4;$ $s_1, s_2 \geq 0$

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Continued…

Table 1

		c_j	-4	-6	-5	1	0	0
c_B	v_B	b	v_1	v_2	v_3	v_4	s_1	s_2
-4	v_1	3	1	0	1	0	-1	0
0	s_2	2	0	-1	-1	1	0	1
$z_j - c_j$			0	6	1	-1	4	0

Continued…

Table 2

		c_j	-4	-6	-5	1	0	0
c_B	v_B	b	v_1	v_2	v_3	v_4	s_1	s_2
-4	v_1	3	1	0	1	0	-1	0
1	v_4	2	0	-1	-1	1	0	1
$z_j - c_j$			0	5	0	0	4	1

$v_1 = 3$, $v_2 = v_3 = 0$, $v_4 = 2$, min $w = 10$; $x_1 = 4$, $x_2 = 1$, max $z = 10$

Duality: Example 2

Use duality to solve the LPP:

Max $Z = 2x_1 + 3x_2$

s. t. $-x_1 + 2x_2 \le 4$

$x_1 + x_2 \le 6$

$x_1 + 3x_2 \le 9$

$x_1, x_2 \ge 0$

Continued…

The standard form

Max $w = -4v_1 - 6v_2 - 9v_3 + 0.s_1 + 0.s_2 - M.s_3 - M.s_4$

s. t.

$-v_1 + v_2 + v_3 - s_1 + s_3 = 3$

$2v_2 + v_2 + 3v_3 - s_2 + s_4 = 2$

$v_1, v_2, v_3, s_1, s_2, s_3, s_4 \ge 0$

Continued…

Table 1

		c_j	-4	-6	-9	0	0	-M	-M
c_B	v_B	b	v_1	v_2	v_3	s_1	s_2	s_3	s_4
-M	s_3	2	-1	1	1	-1	0	1	0
-M	s_4	3	2	1	3	0	-1	0	1
$z_j - c_j$			-M+4	-2M+6	-4M+9	M	M	0	0

Continued…

Table 2

		c_j	-4	-6	-9	0	0	-M	-M
c_B	v_B	b	v_1	v_2	v_3	s_1	s_2	s_3	s_4
-M	s_3	1	-5/3	2/3	0	-1	1/3	1	-1/3
-9	v_3	1	2/3	1/3	1	0	-1/3	0	1/3
$z_j - c_j$			5M/3-2	-M/3+3	0	M	-M/3+3	0	4M/3–3

Continued…

Table 3

		c_j	-4	-6	-9	0	0	-M	-M
c_B	v_B	b	v_1	v_2	v_3	s_1	s_2	s_3	s_4
-6	v_2	3/2	-5/2	1	0	-3/2	1/2	3/2	-1/2
-9	v_3	1/2	3/2	0	1	1/2	-1/2	-1/2	1/2
$z_j - c_j$			11/2	0	0	9/2	3/2	M–9/2	M–3/2

$v_1 = 0$, $v_2 = 3/2$, $v_3 = 1/2$, min $w = 27/2$; $x_1 = 9/2$, $x_2 = 3/2$, max $z = 27/2$