Assignment Problems

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Some Examples

- Assign people to project assignments
- Assign jobs to machines
- Assign products to plants
- Assign tasks to time slots

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The Assignment Problem

Consider the problem of assigning *n* assignees to *n* tasks. Only one task can be assigned to an assignee, and each task must be assigned.

There is also a cost associated with assigning an assignee *i* to task *j*, c_{ij}.

The objective is to assign all tasks such that the total cost is minimized.

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Assumptions in AP

- The number of assignees and the number of tasks are the same (denoted by *n*).
- Each assignee is to be assigned to exactly one task.
- Each task is to be assigned to exactly one assignee.
- There is a cost c_{ij} associated with assignee *i* performing task *j*.
- The objective is to determine how all *n* assignments should be made to minimize the total cost.

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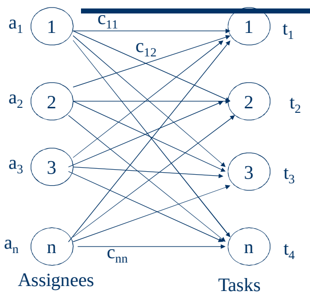
The Flow Diagram



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The Cost Matrix in AP

Let the following represent the standard assignment problem cost matrix, **c**:

		Tasks				
		1	2	…	n	
	1	c_{11}	c_{12}	…	c_{1n}	**1**
Assignees	2	c_{21}	c_{22}	…	c_{2n}	**1**
	…	…				**1**
	n	c_{n1}	c_{n2}	…	c_{nn}	**1**
		1	**1**	**1**	**1**	

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LP Formulation

Minimize $Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$

s.t.

$$\sum_{j=1}^{n} x_{ij} = 1 \ \forall\, i$$

$$\sum_{i=1}^{n} x_{ij} = 1 \ \forall\, j$$

$$x_{ij} \geq 0$$

$$(x_{ij} = 0 \text{ or } 1, \forall\, i \text{ and } j)$$

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The Hungarian Method

Step 1: Find the minimum element in each row. Construct a new matrix by subtracting from each cost the minimum cost in its row. For this new matrix, find the minimum cost in each column. Construct a new matrix by subtracting from each cost the minimum cost in its column.

Step 2: Draw the minimum number of horizontal and/or vertical lines to cover all the zeros of the reduced cost matrix.
(a) If number of lines drawn are equal to the order of the cost matrix, then the optimal assignment is reached. Go to Step 3.
(b) If number of lines drawn are less than order of the cost matrix, then go to Step 4.

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Continued…

- Step 3: Examine all rows containing only one zero. Mark this zero with a box as an assignment will be made there. Perform similar operation in columns. There will be only one zero in each row & each column. Obtain the optimal assignment & corresponding minimum cost.
- Step 4: Find the smallest nonzero element (say, its value is k) in the reduced cost matrix that is uncovered by the lines drawn in Step 2. Subtract *k* from each uncovered element of the matrix and add *k* to each element that is covered by two lines. Return to step 2.

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Example 1

	Job 1	Job 2	Job 3	Job 4
Machine 1	14	**5**	8	7
Machine 2	**2**	12	6	5
Machine 3	7	8	**3**	9
Machine 4	**2**	4	6	10

Row Reduction

	Job 1	Job 2	Job 3	Job 4
Machine 1	9	0	3	2
Machine 2	0	10	4	3
Machine 3	4	5	0	6
Machine 4	0	2	4	8

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Continued…

Column Reduction

	Job 1	Job 2	Job 3	Job 4
Machine 1	9	0	3	0
Machine 2	0	10	4	1
Machine 3	4	5	0	4
Machine 4	0	2	4	6

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Continued…

	Job 1	Job 2	Job 3	Job 4
Machine 1	9	0	3	0
Machine 2	0	10	4	1
Machine 3	4	5	0	4
Machine 4	0	2	4	6

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Continued…

- Find the smallest nonzero element (say, *k*) in the reduced cost matrix that is uncovered by the lines. Subtract *k* from each uncovered element, and add *k* to each element that is covered by two lines.

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Continued…

		Job 1	Job 2	Job 3	Job 4
Machine 1		**10**	0	3	0
Machine 2		0	**9**	**3**	**0**
Machine 3		**5**	5	0	4
Machine 4		0	**1**	**3**	**5**

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Continued…

		Job 1	Job 2	Job 3	Job 4
Machine 1		10	0	3	0
Machine 2		0	**9**	**3**	**0**
Machine 3		5	5	0	4
Machine 4		0	**1**	**3**	**5**

Need 4 lines, so we have the optimal assignment and we stop

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Continued…

		Job 1	Job 2	Job 3	Job 4
Machine 1		10	**0**	3	**0**
Machine 2		**0**	9	3	**0**
Machine 3		5	5	**0**	4
Machine 4		**0**	1	3	5

Optimal assignment

$$x_{12} = 1, x_{33} = 1, x_{41} = 1, x_{24} = 1$$

M1 → J2, M2 → J4, M3 → J3, M4 → J1

Minimum cost = 15 units

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Special Cases

Special Cases

- Number of assignees exceeds the number of tasks:

$$\sum_j x_{ij} \leq 1 \quad \text{for each assignee } i$$

- Number of tasks exceeds the number of assignees:

Add enough dummy assignees to equalize the number of assignees and the number of tasks. The objective function coefficients for these new variable would be zero.

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Continued…

Consider following AP. Convert it to the standard definition of AP.

		Tasks			
		1	2	3	4
	1	10	9	8	7
Assignees	2	4	-	5	6
	3	2	1	8	-

Add "big M" to avoid incompatible assignments, and add a dummy assignee to have equal assignees and tasks.

		Tasks			
		1	2	3	4
	1	10	9	8	7
Assignees	2	4	M	5	6
	3	2	1	8	M
	4	0	0	0	0

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Home Work: 1

	M1	M2	M3	M4
A	5	3	1	8
B	7	9	2	6
C	6	4	5	7
D	5	7	7	6

A → M3, B → M4, C → M2, D → M1, Min Cost = 16 units

Home Work: 2

	M_1	M_2	M_3	M_4
J_1	10	24	30	15
J_2	16	22	28	12
J_3	12	20	32	10
J_4	9	26	34	16

$J_1 \rightarrow M_3$, $J_2 \rightarrow M_2$, $J_3 \rightarrow M_4$, $J_4 \rightarrow M_1$, Min Cost = 71 units

$J_1 \rightarrow M_3$, $J_2 \rightarrow M_4$, $J_3 \rightarrow M_2$, $J_4 \rightarrow M_1$, Min Cost = 71 units

$J_1 \rightarrow M_2$, $J_2 \rightarrow M_3$, $J_3 \rightarrow M_4$, $J_4 \rightarrow M_1$, Min Cost = 71 units